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Thermo-acoustical waves in thermo-plastic materials

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SUMMARY

The propagation velocities and the variation of the amplitudes of thermo-acoustical waves in thermo-plastic materials are theoretically investigated. The constitutive equations of anisotropic thermo-plastic materials are derived from the concept of imaginary decomposition of the deformation rate tensor into the elastic and plastic contributions and from that of the plastic potential. From generalized Vernotte's heat conduction law the propagation condition of the jumps of the velocity gradients and of the temperature rate is obtained. In isotropic materials and in the case of a normal stress vector on the wave front we have two purely mechanical transverse waves and two thermo-longitudinal coupled waves. Formulae for the velocities and amplitudes are quite similar with those for thermo-elastic materials. The variation of the amplitude is discussed. There are, in general, three effects on the variation, that is, the non-planar, heat conduction and plastic flow effects. The transverse waves are subjected only to the non-planar effect, while the thermo-longitudinal waves may grow or decay according to the above three effects.

1. Introduction

The *plasticity theory* has been one of the most important fields of continuum mechanics and until now there has been proposed a myriad variety of that theory. Recently the author presented two kinds of plasticity; one [1] was derived from the hypo-elasticity [2] and the other [3] was due to the concept that there is imaginary decomposition of the deformation rate tensor into the elastic and plastic contributions and the plastic part of it is proportional to the gradient of a plastic yield function. The thermal influence was also taken into consideration to the plasticity [4, 5].

The waves propagating in a material depend on not only their mechanical properties but also on their thermal properties. Therefore a wave, in general, must be a *thermo-mechanical coupled wave.*

In order to cancel the infinite propagation velocity of the thermal disturbance, which is the natural result of the Fourier's law, Vernotte [6] proposed a modified heat conduction law:

$$\dot{q}_i = -\frac{1}{\tau}(q_i + \kappa T_{,i}), \tag{1.1}$$

where q_i and T denote, respectively, the heat flux and the temperature, and τ and κ are, respectively, the relaxation time and the conductivity constant. In this paper a comma followed by a suffix denotes the partial derivative with respect to a coordinate.

Applying (1.1) to the analysis of thermo-mechanical coupled wave propagation, Popov [7], Achenbach [8] and Chen [9] discussed *one*-dimensional wave propagation in *thermo-elastic* materials. Also Tokuoka [10, 11] investigated the propagation and growth and decay of *three*-dimensional waves of *arbitrary form* in *anisotropic* and *isotropic thermo-elastic* materials.

On the other hand pure mechanical waves in generalized Prandtl–Reuss plastic materials were analyzed by Tokuoka [12–14] and recently he applied Vernotte's law (1.1) to thermo-plastic materials introduced in [5], then he investigated the propagation and growth and decay of *plane* coupled waves in those materials [15, 16].

In this paper *three-dimensional thermo-acoustical waves of arbitrary form in thermo-plastic materials* are discussed theoretically. In section 2 the constitutive equations of anisotropic thermo-plastic materials are defined and their isotropic forms are obtained. In section 3 brief summaries of the theory of surface and of the compatibility conditions of the first and second orders are depicted. In section 4 the classification of the waves and the propagation velocities are given and in section 5 the variation of the amplitude is analysed. In the last section six important points of the obtained results are discussed shortly.

2. Thermo-plastic materials

The constitutive equations of the linear thermo-elastic material were derived in [10] as

$$\sigma_{ij}=\rho_0(C_{ijkl}\varepsilon_{kl}+C_{ij}\theta)\,,\qquad \eta=-\frac{1}{T_0}(C_{ij}\varepsilon_{ij}+C\theta)\,, \tag{2.1a,b}$$

where σ_{ij}, η, ρ, ε_{ij} and T denote, respectively, the stress, the specific entropy, the density, the strain and the temperature, and

$$\theta\equiv(T-T_0)/T_0 \tag{2.2}$$

is the dimensionless temperature, and where C_{ijkl}, C_{ij} and C are the material constants and express mechanical, thermo-mechanical coupling and thermal properties of the materials, respectively. Here and henceforth the suffix zero indicates a quantity in the equilibrium state, and the summation convention with repeated suffixes is applied.

2.1. *Anisotropic thermo-plastic materials*

Differentiating (2.1) with respect to time and neglecting small quantities of second-order in the deformation gradient, we have

$$\dot\sigma_{ij}=\rho_0(C_{ijkl}d_{kl}+C_{ij}\dot\theta)\,,\qquad \dot\eta=-\frac{1}{T_0}(C_{ij}d_{ij}+C\dot\theta)\,, \tag{2.3a,b}$$

where

$$d_{ij}=\tfrac{1}{2}(v_{i,j}+v_{j,i}) \tag{2.4}$$

denotes the deformation rate tensor and v_i is the velocity of a material particle.

Now we assume that in a plastic flow state the velocity of a material particle is imaginarily decomposed into elastic and plastic contributions, that is,

$$d_{ij}={}_{\mathrm{E}}d_{ij}+{}_{\mathrm{P}}d_{ij}\,, \tag{2.5}$$

where the suffixes E and P indicate, respectively, the elastic and plastic parts of a quantity. Furthermore we assume that the relations (2.3) hold for materials in plastic flow state when d_{ij} is replaced by ${}_{\mathrm{E}}d_{ij}$. So we have

$$\dot\sigma_{ij}=\rho_0(C_{ijkl}\,{}_{\mathrm{E}}d_{kl}+C_{ij}\dot\theta)\,,\qquad \dot\eta=-\frac{1}{T_0}(C_{ij}\,{}_{\mathrm{E}}d_{ij}+C\dot\theta) \tag{2.6a,b}$$

for thermo-plastic materials.

Here we suppose that plastic flow may occur when the stress state satisfies the *yield condition*:

$$y(\sigma_{ij})=0\,. \tag{2.7}$$

In general the yield function y depends upon a thermodynamic variable, *e.g.*, the temperature. However we assume here that this dependence is so small that it may be neglected and (2.7) holds for the thermo-plastic material.

Von Mises [17] proposed a *plastic flow rule* such that the plastic part of the deformation rate tensor is proportional to the derivative of the *plastic potential* $y(\sigma_{ij})$ with respect to the stress tensor. Then

$${}_{\mathrm{P}}d_{ij}=\phi\,\partial y/\partial\sigma_{ij}\equiv\phi\,y_{ij}\,, \tag{2.8}$$

where ϕ is a scalar proportionality coefficient.
Substituting (2.6a) and

$$_{E}d_{ij} = d_{ij} - \phi y_{ij} \tag{2.9}$$

into the *perfect plasticity condition*:

$$y_{ij}\dot{\sigma}_{ij} = 0 \,, \tag{2.10}$$

we have

$$\phi = F_{ij}d_{ij} + F\dot{\theta} \,, \tag{2.11}$$

where

$$F_{ij} \equiv \frac{y_{ab}C_{abij}}{y_{cd}C_{cdef}y_{ef}} \,, \qquad F \equiv \frac{y_{ab}C_{ab}}{y_{cd}C_{cdef}y_{ef}} \tag{2.12a,b}$$

From (2.6), (2.9) and (2.11) we have the constitutive equations of the anisotropic thermo-plastic materials.

$$\dot{\sigma}_{ij} = P_{ijkl}d_{kl} + P_{ij}\dot{\theta} \,, \qquad \dot{\eta} = P'_{ij}d_{ij} + P\dot{\theta} \,, \tag{2.13a,b}$$

where the plasticity tensors and scalar are given by

$$P_{ijkl} \equiv \rho_0 (C_{ijkl} - C_{ijab}y_{ab}F_{kl}) \,, \tag{2.14a}$$

$$P_{ij} \equiv \rho_0 (C_{ij} - C_{ijab}y_{ab}F) \,, \tag{2.14b}$$

$$P'_{ij} \equiv -\frac{1}{T_0}(C_{ij} - C_{ab}y_{ab}F_{ij}) = -\frac{1}{\rho_0 T_0} P_{ij} \,, \tag{2.14c}$$

$$P \equiv -\frac{1}{T_0}(C - C_{ab}y_{ab}F) \,. \tag{2.14d}$$

Now we assume that the anisotropic heat conduction law [10]:

$$\dot{q}_i = -\nu_{ij}(q_j + K_{jk}\theta_{,k}) \tag{2.15}$$

holds for our thermo-plastic materials, where ν_{ij} and $K_{ij} \equiv \kappa_{ij} T_0$ denote, respectively, the inverse relaxation time tensor and the conductivity tensor.

For the thermo-elastic materials [10] the law of energy conservation is expressed by

$$\rho_0 T_0 \dot{\eta} = -q_{i,i} \,, \tag{2.16}$$

which means that the variation of heat in a portion of the material is due to the heat flow through its boundary surface. Here we assume that (2.16) holds for our thermo-plastic materials*. Then from (2.13b) and (2.16) we have

$$q_{i,i} = -\rho_0 T_0 (P'_{ij}d_{ij} + P\dot{\theta}) \,. \tag{2.17}$$

* We assume that the rate of the specific Helmholtz free energy is expressed as

$$\dot{\psi} = H_{ij}d_{ij} + H\dot{\theta} \,, \tag{F1}$$

and H_{ij} and H may depend upon some state variables but they are independent of the temperature gradient. Then the Clausius–Duhem inequality:

$$\rho(\dot{\psi} + \eta \dot{T}) - \sigma_{ij}d_{ij} + \frac{1}{T} q_i T_{,i} \leqq 0 \tag{F2}$$

is satisfied for all admissible processes, if

$$\sigma_{ij} = \rho H_{ij} \,, \quad \eta = -H/T_0 \,, \quad q_i \theta_{,i} \leqq 0 \,. \tag{F3}$$

From the law of balance of energy, which is expressed as

$$\rho(\psi + \eta T)^{\cdot} = \sigma_{ij}d_{ij} - q_{i,i} + \rho s \,, \tag{F4}$$

we can easily obtain

$$\rho T \dot{\eta} = -q_{i,i} + \rho s \tag{F5}$$

where s denotes the specific heat supply and is assumed to be zero in our thermo-plastic materials.

2.2. *Isotropic thermo-plastic materials*

In an isotropic material material tensors must be isotropic tensors. Hence from the familiar theorems of tensor analysis we have

$$C_{ijkl} = \frac{\lambda}{\rho_0}\delta_{ij}\delta_{kl} + \frac{\mu}{\rho_0}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}), \tag{2.18a}$$

$$C_{ij} = -\frac{3\lambda+2\mu}{\rho_0}\alpha T_0\delta_{ij} \equiv -A\delta_{ij}, \tag{2.18b}$$

$$C = -c_V T_0^2, \tag{2.18c}$$

where λ and μ are the Lamé elastic constants, α is the coefficient of thermal expansion, c_V is the specific heat at constant volume, and δ_{ij} denotes the Kronecker delta. Refer, *e.g.*, to Thomas [18, Chap. 1].

Now we assume the so-called *von Mises yield condition*:

$$\sigma^*_{ij}\sigma^*_{ij} = 2k^2, \tag{2.19}$$

where

$$\sigma^*_{ij} \equiv \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} \tag{2.20}$$

is the deviatoric stress tensor and k is a material constant. This yield condition is a natural consequence derived from the assumptions that $y(\sigma_{ij})$ is a quadratic form of the stress components and its coefficients are isotropic tensors and that $y(\sigma_{ij})$ is pressure-insensitive. See von Mises [17].

Substituting (2.18) and (2.19) into (2.12) and (2.14), we have the isotropic plasticity tensors and scalar:

$$P_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}) - \frac{\mu}{k^2}\sigma^*_{ij}\sigma^*_{kl}, \tag{2.21a}$$

$$P_{ij} = -(3\lambda+2\mu)\alpha T_0\delta_{ij}, \tag{2.21b}$$

$$P'_{ij} = \frac{3\lambda+2\mu}{\rho_0}\alpha\delta_{ij} = -\frac{P_{ij}}{\rho_0 T_0}, \tag{2.21c}$$

$$P = c_V T_0. \tag{2.21d}$$

Thus we have the constitutive equations of an isotropic thermo-plastic material:

$$\dot{\sigma}_{ij} = \lambda d_{kk}\delta_{ij} + 2\mu d_{ij} - \frac{\mu}{k^2}\sigma^*_{kl}d_{kl}\sigma^*_{ij} - (3\lambda+2\mu)\alpha\dot{T}\delta_{ij}, \tag{2.22a}$$

$$\dot{\eta} = \frac{3\lambda+2\mu}{\rho_0}\alpha d_{kk} + c_V\dot{T}, \tag{2.22b}$$

Also we have from (2.17)

$$q_{i,i} = -(3\lambda+2\mu)\alpha T_0 d_{kk} - \rho_0 c_V T_0\dot{T}. \tag{2.23}$$

3. Surfaces of arbitrary form and compatibility conditions of the first and second order

We consider a two-dimensional regular surface Σ in a Euclidian three-dimensional space. It is represented by the form:

$$x_i = \phi_i(\xi^K; t), \tag{3.1}$$

where x_i $(i=1, 2, 3)$ are the Cartesian coordinates of a point on Σ, ξ^K $(K=1, 2)$ are the curvilinear coordinates of the point on Σ and t denotes the time.

Basic formulae are summarized here from the theory of surface. Refer, *e.g.*, to Thomas [18, Chap. 4].

$$n_i n_i = 1\,, \quad n_i x_{i,K} = 0\,, \tag{3.2}$$

$$x_{i,KL} = b_{KL} n_i\,, \quad n_{i,K} = -g^{LM} b_{KL} x_{i,M} \tag{3.3}$$

$$g^{KL} b_{KL} = 2\Omega\,, \tag{3.4}$$

where n_i, $x_{i,K}$, g^{KL}, b_{KL} and Ω denote, respectively, the normal vector, a tangent vector, the contravariant fundamental metric tensor, the second fundamental form and the mean curvature of Σ.

According to Thomas we may introduce an important concept, that is, the δ *time derivative.* From his analysis [19] we have

$$\frac{\delta n_i}{\delta t} = -g^{KL} U_{,K} x_{i,L}\,, \tag{3.5}$$

where U denotes the normal velocity of Σ. Formula (3.5) shows that, if U is constant on Σ at an instant, the normal direction of Σ can not rotate at that time.

If the normal velocity U is homogeneous in the region into which Σ propagates, any surface of Σ at any time is parallel with each other. Thomas [18, Chap. 4] proved that the mean curvature of the parallel surfaces is given by

$$\Omega = \frac{\Omega_0 - K_0 l}{1 - 2\Omega_0 l + K_0 l^2}\,, \tag{3.6}$$

where Ω_0 and K_0 denote, respectively, the mean and Gaussian curvatures of a surface from which the normal distance l is measured.

Now we assume that Σ is a moving surface over which the derivatives of some quantity may have jump discontinuities. We define the discontinuity $[f]$ in f by writing

$$[f] \equiv f_- - f_+\,, \tag{3.7}$$

where the subscripts $-$ and $+$ refer, respectively, to the back and front sides of the moving surface.

The geometrical and kinematical compatibility conditions of the first and second order are given by

$$[f_{,i}] = \bar{f} n_i\,, \quad [\dot{f}] = -U\bar{f}, \tag{3.8a,b}$$

$$[f_{,ij}] = \bar{\bar{f}} n_i n_j + g^{KL} \bar{f}_{,K}(n_i x_{j,L} + n_j x_{i,L}) - \bar{f} g^{KL} g^{MN} b_{KM} x_{i,L} x_{j,N}\,, \tag{3.9a}$$

$$[\dot{f}_{,i}] = \left(-U\bar{\bar{f}} + \frac{\delta \bar{f}}{\delta t}\right) n_i - U g^{KL} x_{i,K} \bar{f}_{,L} \tag{3.9b}$$

$$[\ddot{f}] = U^2 \bar{\bar{f}} - 2U \frac{\delta \bar{f}}{\delta t}\,, \tag{3.9c}$$

where

$$[f] = 0 \tag{3.10}$$

and the constant value of U are assumed and

$$\bar{f} \equiv [f_{,i}] n_i\,, \quad \bar{\bar{f}} \equiv [f_{,ij}] n_i n_j\,. \tag{3.11}$$

See, *e.g.*, Thomas [20, 19] or Truesdell and Toupin [21].

4. Propagation velocities of thermo-acoustical waves in thermo-plastic materials

A moving singular surface associated with the jumps of the velocity gradient and the temperature rate is called a *thermo-acoustical wave* if the following two conditions hold:

(i) *v_i, θ, σ_{ij}, η and q_i are continuous everywhere.*
(ii) *The derivatives of the first and second order of v_i and θ have jump discontinuities across the singular surface but are continuous everywhere else.*

From the relations (2.13) and (2.15) and above definition we can say that the derivatives of the first order of σ_{ij}, η and q_i have jump discontinuities across the singular surface.

The stress and the velocity must satisfy the equation of motion:

$$\sigma_{ij,j} = \rho_0 \dot{v}_i\,, \tag{4.1}$$

where the body force is assumed to be zero.

4.1. *Waves in anisotropic materials*

Applying the definitions (i) and (ii) and the compatibility conditions of the first order (3.8) to (2.13a), (2.17), (4.1) and (2.15) we have

$$-U\bar{\sigma}_{ij} = P_{ijkl}\bar{v}_k n_l - U P_{ij}\bar{\theta}\,, \tag{4.2a}$$

$$\bar{q}_i n_i = -\rho_0 T_0 (P'_{ij}\bar{v}_i n_j - U P\bar{\theta})\,, \tag{4.2b}$$

$$\bar{\sigma}_{ij} n_j = -\rho_0 U \bar{v}_i\,, \tag{4.2c}$$

$$-U\bar{q}_i = -v_{ij} K_{jk} n_k \bar{\theta}\,. \tag{4.2d}$$

Eliminating $\bar{\sigma}_{ij}$ and $\bar{q}_i$ from (4.2), then we have the *propagation conditions* of the thermo-acoustical wave,

$$({}_{\mathrm{P}}Q_{ik} - U^2\delta_{ik})\bar{v}_k - U\,{}_{\mathrm{P}}Q_i\bar{\theta} = 0\,, \tag{4.3a}$$

$$-U\,{}_{\mathrm{P}}Q_i\bar{v}_i + ({}_{\mathrm{P}}Q - U^2 T_0 P)\bar{\theta} = 0\,, \tag{4.3b}$$

where

$${}_{\mathrm{P}}Q_{ik} \equiv \frac{P_{ijkl}n_j n_l}{\rho_0}\,,\quad {}_{\mathrm{P}}Q_i \equiv \frac{P_{ij}n_j}{\rho_0}\,,\quad {}_{\mathrm{P}}Q \equiv \frac{n_i v_{ij} K_{jk} n_k}{\rho_0}\,. \tag{4.4}$$

The propagation conditions (4.3) are combined into a single relation of the form:

$${}_{\mathrm{P}}R_{\alpha\beta}\,a_\beta = 0\,, \tag{4.5}$$

where

$$\left\|{}_{\mathrm{P}}R_{\alpha\beta}\right\| \equiv \begin{Vmatrix} {}_{\mathrm{P}}Q_{ik} - U^2\delta_{ik} & -U\,{}_{\mathrm{P}}Q_i \\ -U\,{}_{\mathrm{P}}Q_i & {}_{\mathrm{P}}Q - U^2 T_0 P \end{Vmatrix}, \tag{4.6}$$

$$a_\alpha \equiv (\bar{v}_i\,, \bar{\theta})\,, \tag{4.7}$$

and the greek suffix runs from one to four.

The propagation velocities of the thermo-acoustical wave are, then, solutions of the equation:

$$\det({}_{\mathrm{P}}R_{\alpha\beta}) = 0\,, \tag{4.8}$$

which yields four roots. So we have, in general, four waves having the same normal direction n_i.

4.2. *Waves in isotropic materials*

Substituting (2.21), $v_{ij} = (1/\tau)\delta_{ij}$ and $K_{ij} = \kappa T_0 \delta_{ij}$ into (4.4) we have

$$\begin{aligned} {}_{\mathrm{P}}Q_{ik} &= (c_{\mathrm{L}}^2 - c_{\mathrm{T}}^2)n_i n_k + c_{\mathrm{T}}^2\delta_{ik} - \frac{c_{\mathrm{T}}^2}{k^2}\sigma_i^*\sigma_k^*\,, \\ {}_{\mathrm{P}}Q_i &= -A n_i\,,\quad {}_{\mathrm{P}}Q = \frac{\kappa T_0}{\rho_0\tau}\,, \end{aligned} \tag{4.9}$$

where

$$c_{\mathrm{L}} \equiv \left(\frac{\lambda + 2\mu}{\rho_0}\right)^{\frac{1}{2}},\quad c_{\mathrm{T}} \equiv \left(\frac{\mu}{\rho_0}\right)^{\frac{1}{2}} \tag{4.10}$$

are, respectively, the longitudinal and transverse wave velocities of the purely mechanical waves in a linear elastic material, and

$$\sigma_i^* \equiv \sigma_{ij}^* n_j \tag{4.11}$$

denotes the deviatoric stress vector acting on the wave front surface.

4.2.1. *Normal stress vector*

If the stress vector (4.11) is normal to the wave front, *i.e.*,

$$\sigma_i^* = \sigma^* n_i\,, \tag{4.12}$$

we have

$${}_{\mathrm{P}}Q_{ik} = ({}_{\mathrm{P}}c_{\mathrm{L}}^2 - c_{\mathrm{T}}^2) n_i n_k + c_{\mathrm{T}}^2 \delta_{ik}\,, \tag{4.13}$$

where

$${}_{\mathrm{P}}c_{\mathrm{L}} \equiv c_{\mathrm{L}} \left(1 - \frac{c_{\mathrm{T}}^2}{c_{\mathrm{L}}^2} \frac{\sigma^{*2}}{k^2}\right)^{\frac{1}{2}}, \tag{4.14}$$

then we have

$$\|{}_{\mathrm{P}}R_{\alpha\beta}\| = \left\| \begin{matrix} c_{\mathrm{T}}^2 - U^2 & 0 & 0 & 0 \\ 0 & c_{\mathrm{T}}^2 - U^2 & 0 & 0 \\ 0 & 0 & {}_{\mathrm{P}}c_{\mathrm{L}}^2 - U^2 & AU \\ 0 & 0 & AU & \dfrac{\kappa T_0}{\rho_0 \tau} - c_{\mathrm{V}} T_0^2 U^2 \end{matrix} \right\|, \tag{4.15}$$

where we adopt $n_i = (0, 0, 1)$.

From the yield condition $\sigma_1^{*3} + \sigma_2^{*2} + \sigma^{*3} = 2k^2$ and the relation $\sigma_1^* + \sigma_2^* + \sigma_3^* = 0$, where σ_1^* and σ_2^* are other two principal deviatoric stresses, we can easily obtain $2/\sqrt{3} \geqq \sigma^*/k \geqq 1$. Then we have

$$c_{\mathrm{L}} \geqq {}_{\mathrm{P}}c_{\mathrm{L}} \geqq c_{\mathrm{L}} \left(\frac{3\lambda + 2\mu}{3(\lambda + 2\mu)}\right)^{\frac{1}{2}}. \tag{4.16}$$

From (4.5) and (4.15) we can say that *two transverse waves are purely mechanical and have the same propagation velocity* c_{T} *as in the case of isotropic linear elastic materials while two other coupling waves, called the thermo-longitudinal waves, have velocities, which are roots of*

$$({}_{\mathrm{P}}c_{\mathrm{L}}^2 - U^2)\left(\frac{\kappa T_0}{\rho_0 \tau} - c_{\mathrm{V}} T_0^2 U^2\right) = A^2 U^2 \tag{4.17}$$

or

$$\left(\frac{U}{{}_{\mathrm{P}}c_{\mathrm{L}}}\right)^4 - (1 + {}_{\mathrm{P}}\beta^2 + {}_{\mathrm{P}}\gamma)\left(\frac{U}{{}_{\mathrm{P}}c_{\mathrm{L}}}\right)^2 + {}_{\mathrm{P}}\beta^2 = 0\,, \tag{4.18}$$

and the ratio of the amplitudes is given by

$$\frac{a_4}{a_3} = \frac{{}_{\mathrm{P}}c_{\mathrm{L}}}{A} \frac{\left(\dfrac{U}{{}_{\mathrm{P}}c_{\mathrm{L}}}\right)^2 - 1}{\left(\dfrac{U}{{}_{\mathrm{P}}c_{\mathrm{L}}}\right)}, \tag{4.19}$$

where

$${}_{\mathrm{P}}\beta \equiv \left(\frac{\kappa}{{}_{\mathrm{P}}c_{\mathrm{L}}^2 \rho_0 \tau c_{\mathrm{V}} T_0}\right)^{\frac{1}{2}}, \quad {}_{\mathrm{P}}\gamma \equiv \frac{A^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^2 c_{\mathrm{V}} T_0^2} \tag{4.20a,b}$$

are dimensionless quantities, which depend not only upon the material constants but also upon the normal deviatoric stress vector acting on the wave front.

Equations (4.17) and (4.18) have the same form as for thermo-longitudinal waves in linear thermo-elastic materials [10, Eqs. (5.7), (5.12)], and we may obtain (4.18) and (4.19) when c_L, β and γ in (5.7) and (5.12) of [10] are, respectively, replaced by ${}_Pc_L$, ${}_P\beta$ and ${}_P\gamma$. Furthermore Figures 1 and 2 of [10] may be used after the same replacements of the three quantities c_L, β and γ.

4.2.2. *Non-normal stress vector*

In this case we can not separate the transverse waves from the thermo-longitudinal waves. If we adopt the x_2-axis such that the (x_2, x_3)-plane contains the stress vector acting on the wave front, we have

$$\|{}_PR_{\alpha\beta}\| = \left\| \begin{matrix} c_T^2-U^2 & 0 & 0 & 0 \\ 0 & c_T^2-U^2 & -\frac{c_T^2}{k^2}\sigma_2^*\sigma^* & 0 \\ 0 & -\frac{c_T^2}{k^2}\sigma_2^*\sigma^* & {}_Pc_L^2-U^2 & AU \\ 0 & 0 & AU & \frac{\kappa T_0}{\rho_0\tau}-c_V T_0^2 U^2 \end{matrix} \right\| . \tag{4.21}$$

The thermo-acoustical tensor (4.21) shows that there is one purely mechanical transverse wave, which has velocity c_T and polarization direction along the x_1-axis, while the other three waves are neither purely mechanical and thermal, nor purely transverse and longitudinal. Henceforth we will focus our attention to the case of a normal stress vector.

5. Variation of amplitudes of thermo-acoustical waves of arbitrary form in isotropic thermo-plastic materials

From the definition (3.7) of the jump discontinuity, the product of two quantities f and g has the following jump discontinuities:

$$[fg]=[f][g]+[f]g_+ +f_+[g], \tag{5.1a}$$

$$[fg]=-[f][g]+[f]g_- +f_-[g]. \tag{5.1b}$$

In this section we consider two kinds of waves, that is, proceeding and receding waves. When the plastic region proceeds into the elastic region we say that the boundary is a *proceeding wave*, while, if the reverse occurs, we say that the boundary is a *receding wave*. Also we assume here that the stresses in both sides satisfy the yield condition and so the stresses have the same values while one side remains in the elastic state such that $v_{i,j}=\dot{\theta}=0$ but other side is in plastic flow. Then, assuming that f and g in (5.1) are the velocity gradient and/or the temperature rate and regarding the boundary as a singular surface, we have

$$[fg]=\pm[f][g], \tag{5.2}$$

where the plus and minus signs refer, respectively, to the proceeding and receding waves.

Now we assume that the deviatoric stress vector σ_i^* given by (4.11) is normal to the wave front and holds a constant value along the path and the waves have constant propagation velocities. Hence we have the case mentioned in the subsection 4.2.1.

Differentiating (4.1) with time, eliminating $\dot{\sigma}_{ij}$ from it and (2.22a), and dividing the result by ρ_0, we have

$$(c_L^2-c_T^2)v_{k,ik}+c_T^2 v_{i,kk}-\ddot{v}_i-A\dot{\theta}_{,i}$$
$$-\frac{c_T^2}{k^2}(\sigma_{kl}^*\sigma_{ij}^* v_{k,lj}+\sigma_{kl,j}^*\sigma_{ij}^* d_{kl}+\sigma_{kl}^*\sigma_{ij,j}^* d_{kl})=0 \tag{5.3}$$

and differentiating (1.1) and (2.23) with x_i and time, respectively, and eliminating $\dot{q}_i$ and q_i from them, and dividing the result by ρ_0, we have

$$A\dot{v}_{k,k}+c_V T_0^2\ddot{\theta} - \frac{\kappa T_0}{\rho_0\tau}\theta_{,kk} + \frac{A}{\tau}v_{k,k} + \frac{c_V T_0^2}{\tau}\dot{\theta} = 0\,. \tag{5.4}$$

Applying the compatibility conditions of the second order (3.9) to (5.3) and (5.4) and using (5.2) and

$$\overline{\sigma_{ij}^*} = -\frac{\mu}{U}\left(a_i n_j + a_j n_i - \tfrac{2}{3}a_k n_k \delta_{ij} - \frac{1}{k^2}\sigma_{kl}^*\sigma_{ij}^* a_k n_l\right), \tag{5.5}$$

we have after some lengthy manipulations that

$$\begin{aligned}
&2U\frac{\delta a_i}{\delta t} - 2\Omega(c_L^2-c_T^2)(a_k n_k)n_i - 2\Omega c_T^2 a_i + 2\Omega\frac{c_T^2}{k^2}(\sigma_k^* a_k)\sigma_i^* \\
&+(c_L^2-c_T^2)g^{KL}\{(a_k x_{k,L})_{,K}n_i + (a_k n_k)_{,K}x_{i,L}\} \\
&-\frac{c_T^2}{k^2}g^{KL}\{\sigma_{kl}^*(a_k n_l)_{,K}\sigma_{ij}^* x_{j,L} + \sigma_{kl}^*(a_k x_{l,L})_{,K}\sigma_i^*\} \\
&\pm\frac{\rho_0 c_T^4}{Uk^2}\left\{\sigma_i^*\left(a_k^2 + \frac{1}{3}(a_k n_k)^2 - \frac{1}{k^2}(\sigma_k^* a_k)^2\right) + (\sigma_k^* a_k)\left(a_i + \tfrac{1}{3}(a_m n_m)n_i - \frac{1}{k^2}\sigma_i^*(\sigma_m^* a_m)\right)\right\} \\
&-A\left(\frac{\delta a_4}{\delta t}n_i - Ug^{KL}x_{i,K}a_{4,L}\right) = \\
&-(c_L^2-c_T^2)(\bar{\bar{v}}_k n_k)n_i + (U^2-c_T^2)\bar{\bar{v}}_i + \frac{c^2}{k^2}(\sigma_k^*\bar{\bar{v}}_k)\sigma_i^* - AU\bar{\bar{\theta}}n_i\,,
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
&A\frac{\delta a_k}{\delta t}n_k + A\left\{\left(2\Omega U + \frac{1}{\tau}\right)(a_k n_k) - g^{KL}U(a_k x_{k,L})_{,K}\right\} \\
&\qquad -2c_V T_0^2 U\frac{\delta a_4}{\delta t} + \left(\frac{2\kappa T_0\Omega}{\rho_0\tau} - \frac{c_V T_0^2}{\tau}U\right)a_4 \\
&= AU(\bar{\bar{v}}_k n_k) + \left(\frac{\kappa T_0}{\rho_0\tau} - c_V T_0^2 U^2\right)\bar{\bar{\theta}}\,.
\end{aligned} \tag{5.7}$$

5.1. *Transverse waves*

For a transverse wave polarized along a tangent direction defined by a unit vector t_i we have the amplitude $a_T \equiv a_k t_k$ and

$$n_i t_i = 0, \quad a_k n_k = a_4 = 0, \quad U = c_T\,. \tag{5.8}$$

Multiplying (5.6) by t_i and referring to (5.8) and (4.12) we can easily obtain

$$\frac{da_T}{dl} = \Omega a_T\,, \tag{5.9}$$

where l denotes the normal path length and $\delta/\delta t = U d/dl$ is used.

From (5.9) and (3.6) we have the global variation formula:

$$a_T(l) = \frac{a_T(0)}{(1-2\Omega_0 l + K_0 l^2)^{\frac{1}{2}}} \tag{5.10}$$

The above formula is identical with that of the thermo-elastic materials.

5.2. *Thermo-longitudinal waves*

For a thermo-longitudinal wave we have the amplitudes $a_3 = a_k n_k$ and a_4, which are connected by (4.19). The propagation velocity U satisfies (4.17) and we have

$$n_k x_{k,K} = a_k x_{k,K} = \sigma^*_{kl} n_k x_{l,K} = \sigma^*_{kl} a_k x_{l,K} = 0. \tag{5.11}$$

Multiplying (5.6) by n_i and referring to (5.11) we have

$$2U\frac{\delta a_3}{\delta t} - 2\Omega\, {}_{\mathrm{P}}c_{\mathrm{L}}^2 a_3 - A\frac{\delta a_4}{\delta t} \pm \frac{2\rho_0 c_{\mathrm{T}}^4}{Uk^2}\left(\frac{4}{3} - \frac{\sigma^{*2}}{k^2}\right)\sigma^* a_3^2$$
$$= (U^2 - {}_{\mathrm{P}}c_{\mathrm{L}}^2)(\bar{\bar{v}}_k n_k) - AU\bar{\bar{\theta}}\,. \tag{5.12}$$

Also multiplying (5.12) and (5.7) by AU and $({}_{\mathrm{P}}c_{\mathrm{L}}^2 - U^2)$, respectively, and referring to (5.11) and (4.17) we have

$$A({}_{\mathrm{P}}c_{\mathrm{L}}^2 - U^2)\frac{\delta a_3}{\delta t} - \{A^2 U + 2c_{\mathrm{V}} T_0^2 U({}_{\mathrm{P}}c_{\mathrm{L}}^2 - U^2)\}\frac{\delta a_4}{\delta t} - \left\{2\Omega A U^3 a_3 - \frac{A}{\tau}({}_{\mathrm{P}}c_{\mathrm{L}}^2 - U^2)\right\}a_3$$
$$+ ({}_{\mathrm{P}}c_{\mathrm{L}}^2 - U^2)\left(\frac{2\kappa T_0 \Omega}{\rho_0 \tau} - \frac{c_{\mathrm{V}} T_0^2 U}{\tau}\right)a_4 \pm \frac{2A\rho_0 c_{\mathrm{T}}^4}{k^2}\left(\frac{4}{3} - \frac{\sigma^{*2}}{k^2}\right)\sigma^* a_3^2 = 0\,. \tag{5.13}$$

Then, expressing a_4 by a_3 from (4.19), using (4.17), (4.20) and $\delta/\delta t = Ud/dl$, and dividing the both sides of (5.13) by

$$(2c_{\mathrm{V}} T_0^2 U {}_{\mathrm{P}}c_{\mathrm{L}}^4 / A)\{{}_{\mathrm{P}}\gamma + (U^2/{}_{\mathrm{P}}c_{\mathrm{L}}^2 - 1)^2\},$$

we can obtain the differential equation:

$$\frac{da_3}{dl} = \left(\Omega - \frac{{}_{\mathrm{P}}\nu}{U\tau}\right)a_3 \mp \Pi_3 a_3^2\,, \tag{5.14}$$

where

$${}_{\mathrm{P}}\nu \equiv \frac{{}_{\mathrm{P}}\beta^2}{2}\frac{\left(\dfrac{U^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^2} - 1\right)^2}{\left(\dfrac{U^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^2}\right)\left\{{}_{\mathrm{P}}\gamma + \left(\dfrac{U^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^2} - 1\right)^2\right\}} \tag{5.15}$$

is a dimensionless quantity and

$$\Pi_3 \equiv {}_{\mathrm{P}}\gamma\left(\frac{c_{\mathrm{T}}^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^3}\right)\left(\frac{\mu}{k}\right)\frac{\left(\dfrac{\sigma^*}{k}\right)\left(\dfrac{4}{3} - \dfrac{\sigma^{*2}}{k^2}\right)}{\left(\dfrac{U}{{}_{\mathrm{P}}c_{\mathrm{L}}}\right)\left\{{}_{\mathrm{P}}\gamma + \left(\dfrac{U^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^2} - 1\right)^2\right\}} \tag{5.16}$$

has the dimension of the inverse of velocity. On the other hand if we express a_3 by a_4 from (4.19) we have

$$\frac{da_4}{dl} = \left(\Omega - \frac{{}_{\mathrm{P}}\nu}{U\tau}\right)a_4 \mp \Pi_4 a_4^2\,, \tag{5.17}$$

where $$\Pi_4 \equiv \alpha T_0\, {}_{\mathrm{P}}\gamma\left(\frac{c_{\mathrm{T}}}{{}_{\mathrm{P}}c_{\mathrm{L}}}\right)^4\left(\frac{3\lambda + 2\mu}{k}\right)\frac{\left(\dfrac{\sigma^*}{k}\right)\left(\dfrac{4}{3} - \dfrac{\sigma^{*2}}{k^2}\right)}{\left(\dfrac{U^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^2} - 1\right)\left\{{}_{\mathrm{P}}\gamma + \left(\dfrac{U^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^2} - 1\right)^2\right\}} \tag{5.18}$$

is a dimensionless quantity.

The quantity ${}_{\mathrm{P}}\nu$, called the *damping factor*, has the same form as that in the case of thermo-longitudinal waves in thermo-elastic materials and we may obtain ${}_{\mathrm{P}}\nu$ by the replacement of c_{L} and β in [11, Eq. (4.9)] by ${}_{\mathrm{P}}c_{\mathrm{L}}$ and ${}_{\mathrm{P}}\beta$, respectively. Figure 3 of [10] is then useful for this case by the above replacement.

The differential equations (5.14) and (5.17) can be integrated easily and we have the global variation formulae:

$$a_\Gamma(l) = \frac{a_\Gamma(0)}{(1-2\Omega_0 l + K_0 l^2)^{\frac{1}{2}}} \frac{\exp\left(-\frac{{}_{\mathrm{P}}\nu}{U\tau} l\right)}{1 \pm \Pi_\Gamma a_\Gamma(0) I(l)} \quad (\Gamma=3,4), \tag{5.19}$$

where $a_3(0)$ and $a_4(0)$ are, respectively, the initial amplitudes of a_3 and a_4 at $l=0$, and

$$I(l) \equiv \int_0^l \frac{\exp\left(-\frac{{}_{\mathrm{P}}\nu}{U\tau} s\right)}{(1-2\Omega_0 s + K_0 s^2)^{\frac{1}{2}}} ds. \tag{5.20}$$

6. Discussions

6.1. *Classification of physical properties of waves*

From the definition (3.7) of the jump we can say that, if $[\dot{\rho}]>0$, $[\dot{\rho}]=0$ and $[\dot{\rho}]<0$ over the wave front, the waves are, respectively, *compressive, equivoluminal* and *expansive waves.* The law of conservation of mass shows $[\dot{\rho}]=-\rho a_k n_k$, then we can say that *the transverse wave is equivoluminal while the thermo-longitudinal wave is an expansive wave if $a_3>0$ and a compressive wave if $a_3<0$.*

The inequalities $[\dot{\theta}]>0$ and $[\dot{\theta}]<0$ denote, respectively, *heating* and *cooling waves.* Then from $[\dot{\theta}]=-Ua_4$ and $U>0$ we can say that, *the wave is cooling if $a_4>0$, while the wave is heating if $a_4<0$.* On the other hand $[\dot{\eta}]>0$ and $[\dot{\eta}]<0$ denote, respectively, *entropy increasing* and *decreasing waves.*

Figures 1 and 2 of [10], which hold for our case when c_{L}, β and γ are replaced, respectively, by ${}_{\mathrm{P}}c_{\mathrm{L}}$, ${}_{\mathrm{P}}\beta$ and ${}_{\mathrm{P}}\gamma$, show that there are two kinds of waves, *i.e.*, the *almost mechanical faster wave* for $U>{}_{\mathrm{P}}c_{\mathrm{L}}$ and the *almost temperature rate slower wave* for $U<{}_{\mathrm{P}}c_{\mathrm{L}}$.*

From (2.22b), (4.18) and (4.19) we can easily obtain that

$$[\dot{\eta}] = \frac{{}_{\mathrm{P}}\beta^2 c_{\mathrm{V}} T_0}{\left(\frac{U}{{}_{\mathrm{P}}c_{\mathrm{L}}}\right)^2}[\dot{\theta}] = \frac{{}_{\mathrm{P}}\beta^2\, {}_{\mathrm{P}}c_{\mathrm{L}}^2 c_{\mathrm{V}} T_0}{\rho_0 A} \frac{\left(\frac{U^2}{{}_{\mathrm{P}}c_{\mathrm{L}}^2}-1\right)}{\left(\frac{U}{{}_{\mathrm{P}}c_{\mathrm{L}}}\right)^2}[\dot{\rho}]. \tag{6.1}$$

Then we have formally four kinds of thermo-longitudinal waves:

(i) *The entropy increasing, heating, compressive, faster and almost mechanical wave,*
(ii) *The entropy increasing, heating, expansive, slower and almost temperature rate wave,*
(iii) *The entropy decreasing, cooling, expansive, faster and almost mechanical wave,*
(iv) *The entropy decreasing, cooling, compressive, slower and almost temperature rate wave.*

6.2. *Effects on variation of amplitude*

There are three kinds of effects on the variation of amplitudes of the waves, that is, (i) *The non-planar effect,* (ii) *The heat conduction effect,* and (iii) *The plastic flow effects.*

* We now exclude the special case ${}_{\mathrm{P}}\gamma=0$. In this case a wave having $U={}_{\mathrm{P}}c_{\mathrm{L}}$ exists.

6.2.1. *The non-planar effect*

The factor Ω in (5.9), (5.14) and (5.17), and the factor $(1-2\Omega_0 l+K_0 l^2)^{\frac{1}{2}}$ in (5.10), (5.19) and (5.20) show the non-planar effect. From (5.10) we can say that *the variation of the amplitude of a transverse wave is subjected only to the non-planar effect.* If we have a positive real root l_s of

$$1-2\Omega_0 l_s+K_0 l_s^2=0\,, \tag{6.2}$$

the wave front has a focal point or a focal line at $l=l_s$.*

6.2.2. *The heat conduction effect*

The positive dimensionless quantity ${}_P\beta$ of (4.20a) depends upon the heat conduction, which is represented by a non-zero value of κ. Then the damping factor ${}_P\nu$ of (5.15) depends upon the heat conduction. Formulae (5.19) show that the heat conduction damps the amplitudes of the waves exponentially with respect to the path length.

6.2.3. *The plastic flow effect*

If the material concerned is not a plastic material but an elastic material, the yield strength k must tend to infinity. Then the quantities Π_Γ $(\Gamma=3,4)$ of (5.16) and (5.18) reduce to zero, and we have the differential equations:

$$\frac{da_\Gamma}{dl}=\left(\Omega-\frac{{}_P\nu}{U\tau}\right)a_\Gamma \quad (\Gamma=3,4) \tag{6.3}$$

These are, of course, identical with (4.8) of [11]. Hence we can say that the quantities Π_Γ indicate the plastic flow effect.

6.3. *General criteria of growth and decay of wave*

The integral $I(l)$ of (5.20) is a monotonically increasing function of l and $I(0)=0$. Then the formulae (5.19) show that (a) if $\pm\Pi_\Gamma s_\Gamma(0)$ are non-negative, the amplitudes $a_\Gamma(l)$ decrease monotonically, while (b) if $\pm\Pi_\Gamma a_\Gamma$ are negative and if the path length l_∞, which is a smaller solution of

$$I(l_\infty)=\mp\frac{1}{\Pi_\Gamma a_\Gamma(0)}\,, \tag{6.4}$$

exist, the $a_\Gamma(l)$ may blast out at $l=l_\infty$, and (c) if $\pm\Pi_\Gamma a_\Gamma<0$ and if there is no solution of (6.4), $a_\Gamma(l)$ cannot blast out.

At $l=l_\infty$ we have an infinite magnitude of the amplitudes a_Γ. Then the definition (i) of the thermo-acoustical wave depicted in section 4 can not be applied at the point and we may have discontinuities of v_i and θ there. So we can say that the thermo-acoustical waves may coalesce into *thermo-mechanical shocks* when the amplitudes blast out.

From (4.19) the amplitude ratio is independent of the path length, then if a_3 or a_4 blasts out at a point, the other must do at the same point, and if one of them does not blast out, the other does not also. Thus the above discussion shows that a_3 and a_4 must have the same sign. Certainly from (4.19), (5.16) and (5.18) we have

$$\Pi_3 a_3=\Pi_4 a_4\,. \tag{6.5}$$

Then we can say from (6.4) and 6.5) that the amplitudes of the mechanical and thermal components of the thermo-longitudinal wave, if they blast out, tend to infinity at the same time. Therefore among the four kinds of waves formally given in subsection 6.1., we have two ad-

* If two positive real roots of (6.2) exist, l_s must be taken to be the smaller root.

missible waves (i) and (iii). The upper and lower signs of the double signs in (5.19) refer, respectively, to proceeding and receding waves. Hence we can say that *the proceeding compressive heating and the receding expansive cooling thermo-acoustical waves may blast out and coalesce into shock waves if* (6.4) *has a positive solution.* These two waves may be classified into a same type of wave, that is, the boundary wave where the material in the plastic flow side is more compressive and heating than that in the elastic side.

6.4. *Plane waves*

In this case $\Omega_0 = K_0 = 0$, (5.20) can be integrated and we have

$$a_\Gamma(l) = \frac{a_\Gamma(0)\exp\left(-\frac{\mathrm{p}^\nu}{U\tau}l\right)}{1 \pm \frac{a_\Gamma(0)}{\Lambda_\Gamma}\left(1-\exp\left(-\frac{\mathrm{p}^\nu}{U\tau}l\right)\right)} \qquad (\Gamma=3, 4), \tag{6.6}$$

where

$$\Lambda_\Gamma \equiv \frac{\mathrm{p}^\nu}{U\tau\Pi_\Gamma} \qquad (\Gamma=3, 4) \tag{6.7}$$

are called the *critical amplitudes.* The formula (6.6) is identical with the one-dimensional acceleration wave in materials with memory analyzed by Coleman and Gurtin [22].

Then we can say that

If either

(a) *the absolute magnitudes of the initial amplitudes are less than the critical amplitude, or*

(b) *the proceeding expansive heating wave, or receding compressive cooling wave is concerned, then the amplitudes decay down monotonically as the path length tends to infinity.*

If the initial amplitudes are equal to the critical amplitudes, then the amplitudes remain constant values.

On the other hand, if both

(c) *the absolute magnitudes of the initial amplitudes are larger than the critical amplitudes, and*

(d) *the proceeding compressive heating wave, or the receding expansive cooling wave is concerned,*

then the amplitudes blast out monotonically in the finite path length given by

$$l_\infty = -\frac{U\tau}{\mathrm{p}^\nu}\log\left(1 \pm \frac{\Lambda_\Gamma}{a_\Gamma(0)}\right). \tag{6.8}$$

6.5. *The case of no plastic flow effect*

If $\Pi_\Gamma = 0$, the differential equations (5.14) and (5.17), and the formulae (5.19) reduce to those for thermo-elastic materials, which were analyzed in the former article [11].

From (5.16) and (5.18) we have $\Pi_\Gamma = 0$, if $\sigma^* = 0$ or $\pm 2k/\sqrt{3}$. It is easily obtained that these stress states are the following cases:

$$\sigma_1^* = -\sigma_2^* = \pm k, \quad \sigma^* = 0, \tag{6.9}$$

and

$$\sigma_1^* = \sigma_2^* = \mp\frac{k}{\sqrt{3}}, \quad \sigma^* = \frac{2}{\sqrt{3}}k, \tag{6.10}$$

where σ_1^* and σ_2^* are two principal deviatoric stresses whose principal axes are tangent to the wave front.

6.6. *Non-conductor and a material with no thermo-mechanical coupling*

A material, in which heat can not flow, is called a *non-conductor*. The non-conductor has a vanishing value of the conductivity $K_{ij}=\kappa=0$, thus ${}_{\mathrm{P}}Q=0$ from $(4.4)_3$. Then from (4.5), (4.6) and (4.8) a thermally discontinuous static surface may exist, where we have $a_4\neq 0$ and $a_i= U=0$.

A material with no thermo-mechanical coupling has the constitutive equations

$$\mathring{\sigma}_{ij}=P_{ijkl}d_{kl},\quad \dot{\eta}=P\dot{\theta}, \tag{6.11a,b}$$

i.e., they are separated into purely mechanical and purely thermal equations. Taking $P_{ij}=\alpha=0$ in (4.4) we have ${}_{\mathrm{P}}Q_i=0$. Then the thermo-acoustical tensor ${}_{\mathrm{P}}R_{\alpha\beta}$ is separated into two parts and the propagation condition (4.8) reduces to

$$\det({}_{\mathrm{P}}Q_{ik}-U^2\delta_{ik})=0,\quad U^2=\frac{{}_{\mathrm{P}}Q}{T_0P}. \tag{6.12a,b}$$

Equations (6.12a) and (6.12b) give, respectively, the propagation velocities of the purely mechanical acoustical waves and of the pure temperature rate wave.

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